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Abstract

The Heston model is one of the most popular stochastic volatility models for Equity and FX modelling. Although it was developed more than fifteen years ago, its understanding is still not complete and many recent publications have addressed deep theoretical and implementation issues. We review here some recent results on this model up to and including year 2010.

Introduction

The Heston model is one of the most popular stochastic volatility models for Equity and FX derivatives pricing. Even if the pricing of European vanilla options within this model has been settled for more than fifteen years from the theoretical viewpoint, a bunch of recent papers address the issue of computing effectively the Heston Call/Put semi-analytical formula. Another front where the pace of innovation has been accelerating very recently is the design of simulation algorithms for stochastic volatility models, the classical Euler approximation scheme being notoriously deficient in general for such models. In fact even the mathematical understanding of the Heston model is not complete: recent works suggest that the 2 regimes $\kappa - \rho\sigma > 0$ and $\kappa - \rho\sigma \leq 0$ behave very differently (the parameters are defined in (1)). Only the first regime, which covers the usual case of negative stock/volatility correlation, seems to be completely understood so far.

The first issue raised a couple of years ago was about the pricing of European vanilla options. Although the semi-closed formula has been known since the beginning (see [37] in 1993), it is not straightforward to implement because of the potential oscillatory behaviour of the complex integrand which comes into play through the Fourier-type inversion formula. Fast Fourier Transform methods often work well for a given range of the model parameters, but show poor robustness properties when the model parameters move—which is almost systematically the case in a calibration procedure, for instance. The rigorous and efficient computation requires a thorough study of the integrals involved as well as a proper handling of complex numbers and branch cuts. Several solutions have been proposed in the literature, which suggest a robust implementation.

Similarly, standard and traditional Euler schemes for Monte Carlo simulations were proved to be insufficient: the price to be computed is very sensitive to the effective value of the volatility which is too coarsely approximated by the Euler scheme. More advanced techniques, adapted to Heston-like models (square-root processes) have recently been developed to handle this issue properly and efficiently.

From a more theoretical point of view, some properties have recently been proved, making both the understanding and the use of the Heston model much easier, even if some issues remain. In particular, the question of the true martingality of an asset following Heston dynamics was not clear until recently: indeed, in case of positive stock-volatility correlation, the stock price can grow way too fast, and many popular stochastic models are not true martingales but only local martingales (in other words, *strict* local martingales), as studied for instance by Jourdain in [44].

Furthermore, symmetries as well as particular combinations of the model parameters have been revealed, thus casting a new light upon the practical meaning (and influence) of these parameters.

Finally, asymptotic properties of the model have been thoroughly studied: small and large time behaviours, wings of the implied volatility for fixed maturity. Though theoretical these results might seem at first sight, they are actually extremely useful in practice to determine initial points for the calibration of the model, for instance. On this calibration issue, least-square minimisation methods based on these asymptotical results as a proxy have proved to be amazingly accurate.

All these very recent advances (in the last four years or so) have however left aside a couple of issues still to be solved. One in particular has recently been revealed: the importance of the $\kappa - \rho\sigma > 0$ condition, which turns out to be most of the time hidden in numerous papers. Even if on the equity market, the stock-volatility correlation is almost always negative, on FX markets, the correlation between the exchange rate and the instantaneous volatility is positive in one direction and negative in the other, so that for small κ and a positive correlation the condition could fail. Very few papers have recently tackled this issue and we will mention them, as it should be.

The order of the presentation will follow a natural approach, by first looking at the properties of the Heston model, then by studying the pricing of vanilla options, and finally by reviewing asymptotical results and the recent extensions of the model.

1 The Heston model and its parameters

1.1 The generic Heston model

Consider the forward price process $(F_t)_{t\geq 0}$. Under the risk-neutral measure, the Heston stochastic volatility model [37] takes the following form:

$$\begin{aligned} \mathrm{d}F_t &= F_t\sqrt{v_t}\,\mathrm{d}W_t, & F_0 > 0 \\ \mathrm{d}v_t &= \kappa\left(\theta - v_t\right)\mathrm{d}t + \sigma\sqrt{v_t}\,\mathrm{d}Z_t, & v_0 > 0 \\ \mathrm{d}\left\langle W, Z\right\rangle_t &= \rho\,\mathrm{d}t, \end{aligned} \tag{1}$$

where W and Z are two standard Brownian motions, $\kappa > 0$, $\theta > 0$ and $\sigma > 0$ are respectively the mean-reversion speed, the long-term variance and the instantaneous volatility of the variance process $(v_t)_{t\geq 0}$. Note that $v_0 > 0$ is assumed to be non random at time 0, and accounts for the short-term at-the-money implied variance observed on the market. The parameter $\rho \in [-1, 1]$ represents the correlation between the asset price and its instantaneous volatility, and corresponds to the so-called *leverage effect*. The variance process $(v_t)_{t\geq 0}$ is called a Feller diffusion (or a CIR process) and the Yamada-Watanabe conditions (see [46, Section 5.2.C]) ensure that a non-negative unique strong solution exists. Since the process $(F_t)_{t\geq 0}$ can be written as the exponential of a smooth functional of the variance process, it also has a unique strong solution. Since the square-root function is not smooth at the origin, understanding the behaviour of the variance process at this point is fundamental. The Feller classification of boundaries for one-dimensional diffusions (see [47, Chapter 15, Section 6]) implies the following:

(i) if $2\kappa\theta \geq \sigma^2$, then the origin is unattainable;

(ii) if $2\kappa\theta < \sigma^2$, then the origin is a regular, attainable and reflecting boundary; this means that the variance process can touch 0 in finite time, but does not spend time there;

(iii) infinity is a natural boundary, i.e. it can not be attained in finite time and the process can not be started there.

In the rest of this paper, we shall refer to the inequality $2\kappa\theta \geq \sigma^2$ as the *Feller condition*. We refer the interested reader to [54] for the implications of such a boundary classification when pricing options using finite difference schemes.

Notation 1. *We will from now on denote $F_t \sim \mathcal{H}(F_0, v_0, \kappa, \theta, \sigma, \rho, t)$ the price of an asset following the Heston dynamics* (1)*. Since we consider a forward price, the interest rate, the dividends and the repo rates are already included in the price F_t itself.*

Since we shall be interested in the properties of both the variance process $(v_t)_{t\geq 0}$ and the forward price process $(F_t)_{t\geq 0}$, it is worth recalling the following. For any $\delta \geq 0$ and $y_0 \geq 0$, the unique strong solution to the stochastic differential equation $\mathrm{d}Y_t = \delta \mathrm{d}t + 2\sqrt{X_t}\mathrm{d}W_t$, with $Y_0 = y_0$ and $(W_t)_{t\geq 0}$ being a standard Brownian motion, is called a squared Bessel process with dimension δ. Straightforward calculations show that the variance process $(v_t)_{t\geq 0}$ governing the Heston model can be written as

$$v_t \overset{\text{(in law)}}{=} \mathrm{e}^{-\kappa t} Y_{\sigma^2(\mathrm{e}^{-\kappa t}-1)/(4\kappa)}, \qquad \text{for any } t \geq 0, \tag{2}$$

where $(Y_t)_{t\geq 0}$ is a squared Bessel process starting at v_0 with dimension $4\kappa\theta/\sigma^2$ This in particular implies that, for any $0 \leq s \leq t$, the conditional distribution of v_t knowing v_s is a chi-squared distribution (see [33] for more details).

Notation 2. *In relation to Bessel processes* (2)*, the CIR process in* (1) *is usually written $\mathrm{d}v_t = (\kappa\theta - \kappa v_t)\,\mathrm{d}t + \sigma\sqrt{v_t}\,\mathrm{d}Z_t$, and we therefore define*

$$\overline{\mathcal{H}}(F_0, v_0, \kappa, \kappa\theta, \sigma, \rho, t) := \mathcal{H}(F_0, v_0, \kappa, \theta, \sigma, \rho, t).$$

1.2 The martingale issue: size matters.

Even if stochastic models driven by pure Brownian terms like (1) are always *local* martingales, a size issue could prevent them from being fully qualified martingales, as shown by Jourdain [44]. In the case of the Heston model, this could happen when the stochasticity of the volatility makes the underlying get larger, that is, in case of positive correlation. However, everything turns out to be fine (for the first moment only, as we will see later): The following proposition is proven in [44] by a clever use of the Gronwall's lemma. It was also proved in [54] by a careful study of the Novikov criterion:

Proposition 3. *If $F_t \sim \mathcal{H}(F_0, v_0, \kappa, \theta, \sigma, \rho, t)$ then $(F_t)_{t\geq 0}$ is a true martingale.*

A clear self-contained proof of the true martingality of the Heston model for general parameters has been provided recently by Keller-Ressel in [48] and also in [18], by studying the characteristic transform in the framework of affine processes in relation with solutions of generalised Riccati equations.

1.3 A useful reparameterisation

A time-scaling of the Heston SDE (1) so that the rescaled volatility has a normalised volatility of volatility gives the following property:

$$\mathcal{H}(F_0, v_0, \kappa, \theta, \sigma, \rho, t) = \mathcal{H}\left(F_0, \frac{v_0}{\sigma}, \frac{\kappa}{\sigma}, \frac{\theta}{\sigma}, 1, \rho, \sigma t\right) = \overline{\mathcal{H}}\left(F_0, \bar{v}_0, \alpha, \psi, 1, \rho, \bar{t}\right),$$

with $\bar{v}_0 := v_0/\sigma$, $\alpha := \kappa/\sigma$, $\psi := \kappa\theta/\sigma^2$ and $\bar{t} := \sigma t$. In other words, the Heston dynamics is that of a stochastic volatility model $\overline{\mathcal{H}}$ with unit volatility of volatility parameterised by four parameters: $\bar{v}_0, \alpha, \psi, \rho$. If ψ and α are fixed, the volatility of volatility σ can be seen as a multiplier of the calendar clock at which looking at the model $\overline{\mathcal{H}}$. Many practitioners use this reparameterisation of the Heston model, which is suggested in [18].

1.4 Heston Symmetry and the *large correlation regime*

1.4.1 Inversion and Share measure

In this section,we shall highlight the symmetric structure of the Heston model. Before doing so, let us define the *Share measure* as the probability measure taking the ratio F_T/F_0 as the new numéraire. Note that this is a well-defined transformation since

Proposition 3 above ensures that the process $(F_t)_{t\geq 0}$ is a true martingale. The following lemma conveys a lot of information on the influence of the Heston parameters. Essentially, it says that when one considers an asset $(F_t)_{t\geq 0}$ following some Heston dynamics, then the process $\left(F_t^{-1}\right)_{t\geq 0}$ also follows a Heston dynamics up to the related change of measure.

Lemma 4. *(Heston Symmetry)*
If $F_t \sim \mathcal{H}\left(F_0, v_0, \kappa, \theta, \sigma, \rho, t\right)$, then under the Share measure—i.e., where F_T/F_0 is the new numéraire—we have $F_t^{-1} \sim \mathcal{H}\left(F_0^{-1}, v_0, \widetilde{\kappa}, \widetilde{\theta}, \sigma, -\rho, t\right)$, where $\widetilde{\kappa} := \kappa - \rho\sigma$ and $\widetilde{\theta} := \kappa\theta/\widetilde{\kappa}$. In rescaled parameters under the Share measure this reads $F_t^{-1} \sim \overline{\mathcal{H}}\left(F_0^{-1}, \bar{v}_0, \alpha - \rho, \psi, 1, -\rho, \bar{t}\right)$.

The proof of this mainly relies upon a simple Girsanov-change of measure argument. In [17], the author proves the same but using the characteristic function of Heston. Obviously this Heston symmetry is an involution on the Heston dynamics such that $\kappa - \rho\sigma > 0$, with a single fixed point which is the uncorrelated Heston model ($\rho = 0$). By turning a Call price expectation into a Put price under the Share measure, it can be shown that the right-wing of the Heston model is the left wing of its symmetrised model (and vice-versa). This is useful in practice since many authors provide results on smile asymptotics only on one side of the smile. A last consequence is that the at-the-money volatility structure (aka the implied volatility *term structure*) should be left invariant by the Heston symmetry.

1.4.2 The *large correlation regime*

The previous discussion clearly shows that the region of parameters $\kappa - \rho\sigma \leq 0$, or yet in rescaled parameters $\alpha \leq \rho$, is singular in some sense: the symmetry can not be applied in a straightforward manner since the corresponding volatility parameters $\widetilde{\kappa}$ and $\widetilde{\theta}$ will be negative. Even if some sense could be given to the solution of the stochastic differential equation so obtained, the properties of the volatility process will be quite different than in the usual case. From now on we will call the region $\kappa - \rho\sigma \leq 0$ ($\alpha \leq \rho$) the *large correlation regime*, and its complementary the *good correlation regime*.

1.5 Anticorrelated Heston

The case $\rho = -1$ in Heston (by symmetry the case $\rho = 1$ has analogous properties) has some interesting features. In particular, as stated in the following proposition, the implied volatility is a strictly decreasing function of the strike, and hence the anticorrelated Heston model will is not able to fit smiles that do not have this property.

Proposition 5. *Assume that* $\rho = -1$ *and define* $F^*_{\overline{T}} := F_0 \exp\left(\bar{v}_0 + \psi \overline{T}\right)$ *for all* $\overline{T} = \sigma T > 0$, *then the inequality* $F_{\overline{T}} < F^*_{\overline{T}}$ *always holds. The implied volatility is strictly decreasing for all strikes below* $F^*_{\overline{T}}$ *and null whenever the strike is greater than* $F^*_{\overline{T}}$.

The proof relies on the observation of the system of SDEs (1) in the anti-correlated case. Note that this property is obtained by Fourier transform in Gatheral's book [30]. The anti-correlated Heston is sometimes named the Heston-Nandi model [38], by reference to a work of Heston and Nandi who studied a discrete time version of the anti-correlated Heston.

2 Pricing and calibration

2.1 The semi-closed-form formula

Heston [37] solved the pricing problem for a European Call option $C_t(F_t, v_t)$ at time t, with time to maturity $\tau = T - t$, written on a forward stock price F_t, with strike K:

$$C_t(F_t, v_t) = F_t \mathrm{e}^{-r\tau} P_1(F_t, v_t) - K \mathrm{e}^{-r\tau} P_2(F_t, v_t)$$

with

$$\begin{cases} P_j(F_t, v_t) := \dfrac{1}{2} + \dfrac{1}{\pi} \displaystyle\int_0^{+\infty} \Re\left(\dfrac{f_j(F_t, v_t, x)}{\mathtt{i}x} \mathrm{e}^{-\mathtt{i}xk}\right) \mathrm{d}x \\ f_j(F_t, v_t, x) := \exp\Big(C_j(\tau, x) + D_j(\tau, x) v + \mathtt{i}x \log(F_t)\Big) \\ C_j(\tau, x) := \dfrac{\kappa\theta}{\sigma^2}\left((b_j - \mathtt{i}\rho\sigma x + d_j)\tau - 2\log\left(\dfrac{1 - \gamma_j \exp(d_j \tau)}{1 - \gamma_j}\right)\right) \\ D_j(\tau, x) := \dfrac{b_j - \mathtt{i}\rho\sigma x + d_j}{\sigma^2}\left(\dfrac{1 - \exp(d_j \tau)}{1 - \gamma_j \exp(d_j \tau)}\right) \\ \gamma_j := \dfrac{b_j - \mathtt{i}\rho\sigma x + d_j}{b_j - \mathtt{i}\rho\sigma x - d_j}, \quad d_j := \sqrt{(b_j - \mathtt{i}\rho x)^2 - \sigma^2 x (2\mathtt{i}u_j - x)} \\ u_1 := \dfrac{1}{2}, \quad u_2 := -\dfrac{1}{2}, \quad k := \log(K), \quad b_1 := \kappa - \rho\sigma, \quad b_2 := \kappa \end{cases} \tag{3}$$

So the problem of pricing a Call option in the Heston model eventually boils down to efficiently computing the above complex integral. Several methods are available, either by direct integration (Gauss-Legendre, Gauss-Lobatto) or by FFT. In rescaled parameters, the formula can be simplified as

$$C_t\left(F_t, v_t\right) = F_t \mathrm{e}^{-r\tau} \overline{P}_1\left(F_t, \bar{v}_t\right) - K \mathrm{e}^{-r\tau} \overline{P}_2\left(F_t, \bar{v}_t\right)$$

with

$$\begin{cases} \overline{P}_j\left(F_t, \bar{v}_t\right) := \dfrac{1}{2} + \dfrac{1}{\pi} \displaystyle\int_0^{+\infty} \Re\left(\frac{\bar{f}_j\left(F_t, \bar{v}_t, x\right)}{\mathtt{i}x} \mathrm{e}^{-\mathtt{i}xk}\right) \mathrm{d}x \\ \bar{f}_j\left(F_t, \bar{v}_t, x\right) := \exp\Big(C_j\left(\overline{\tau}, x\right) + D_j\left(\overline{\tau}, x\right)\bar{v} + \mathtt{i}x\log\left(F_t\right)\Big) \\ C_j\left(\overline{\tau}, x\right) := \psi\left(\left(\bar{b}_j - \mathtt{i}\rho x + \bar{d}_j\right)\overline{\tau} - 2\log\left(\dfrac{1-\gamma_j \exp\left(\bar{d}_j\overline{\tau}\right)}{1-\gamma_j}\right)\right) \\ \overline{D}_j\left(\overline{\tau}, x\right) := \left(\bar{b}_j - \mathtt{i}\rho x + \bar{d}_j\right)\left(\dfrac{1-\exp\left(\bar{d}_j\overline{\tau}\right)}{1-\gamma_j\exp\left(\bar{d}_j\overline{\tau}\right)}\right) \\ \overline{\gamma}_j := \gamma_j = \dfrac{\bar{b}_j - \mathtt{i}\rho x + \bar{d}_j}{\bar{b}_j - \mathtt{i}\rho x - \bar{d}_j}, \quad \bar{d}_j := \sqrt{\left(\bar{b}_j - \mathtt{i}\rho x\right)^2 - x\left(2iu_j - x\right)} \\ u_1 := \dfrac{1}{2}, \quad u_2 := -\dfrac{1}{2}, \quad k := \log\left(K\right), \quad \bar{b}_1 := \alpha - \rho, \quad \bar{b}_2 := \alpha, \quad \overline{\tau} := \sigma\tau \end{cases} \tag{4}$$

Note that (3) and (4) hold both in the good and large correlation regimes.

2.2 Jaeckel & Kahl's approach

Consider Equation (3), where we have $P_j\left(F_t, v_t\right) = 1/2 + \pi^{-1}\int_0^{+\infty}\Psi_j\left(x\right)\mathrm{d}x,\ j = 1, 2$. Jaeckel and Kahl [40] proved that the functions Ψ_j have genuine limits both at zero and at infinity, and suggest the use of an adaptive Gauss-Lobatto type integration algorithm after mapping the interval of integration to the compact interval $[0, 1]$ to deal with the oscillatory behaviour of the integrand.

2.3 "The Little Heston trap"

Albrecher et al. [1] proposed to make an efficient transformation when computing the characteristic function of the log forward asset price, defined by $\phi\left(u, t, T\right) = \mathbb{E}_t\left(\mathrm{e}^{\mathrm{i}u\log(F_T)}\right)$. From [40], we know that this is worth (the subscript ϕ_1 is to differentiate it from Albrecher et al.'s one)

$$\phi_1\left(u, t, T\right) = \exp\Big(\mathtt{i}u\log\left(F_t\right) + C\left(\overline{\tau}, u\right) + D\left(\overline{\tau}, u\right)\bar{v}_t\Big),$$

with

$$
\begin{aligned}
C(\overline{\tau}, u) &:= \psi\left(\left(\alpha - \rho u i + \bar{d}_u\right)\overline{\tau} - 2\log\left(\frac{\gamma_u \exp\left(\bar{d}_u \overline{\tau}\right) - 1}{\gamma_u - 1}\right)\right), \\
\overline{D}(\overline{\tau}, u) &:= \left(\alpha - \mathtt{i}\rho u + \bar{d}_u\right)\left(\frac{\exp\left(\bar{d}_u \overline{\tau}\right) - 1}{\gamma_u \exp\left(\bar{d}_u \overline{\tau}\right) - 1}\right), \\
\bar{d}_u &:= \sqrt{(\alpha - \mathtt{i}\rho u)^2 + (u^2 + ui)}, \quad \gamma_u := \frac{\alpha - \mathtt{i}\rho u + \bar{d}_u}{\alpha - \mathtt{i}\rho u - \bar{d}_u}, \quad \overline{\tau} := \sigma(T - t).
\end{aligned}
$$

Due to the branch cuts od the *complex* square root and/or logarithm, a great care is needed while handling such expressions. We refer to [1], section 3, for a detailed explanation of the fact that switching from γ_u to γ_u^{-1} prevents from discontinuity along the evil real-negative axis. The transformed characteristic function is then :

$$
\phi_2(u, t, T) = \exp\left(\mathtt{i}u \log(F_t) + \widetilde{C}(\overline{\tau}, u) + \widetilde{D}(\overline{\tau}, u)\,\bar{v}_t\right),
$$

with

$$
\begin{aligned}
\widetilde{C}(\overline{\tau}, u) &:= \psi\left(\left(\alpha - \mathtt{i}\rho u - \bar{d}_u\right)\overline{\tau} - 2\log\left(\frac{\widetilde{\gamma}_u \exp\left(-\bar{d}_u \overline{\tau}\right) - 1}{\widetilde{\gamma}_u - 1}\right)\right), \\
\widetilde{D}(\overline{\tau}, u) &:= \left(\alpha - \mathtt{i}\rho u - \bar{d}_u\right)\left(\frac{\exp\left(-\bar{d}_u \overline{\tau}\right) - 1}{\widetilde{\gamma}_u \exp\left(-\bar{d}_u \overline{\tau}\right) - 1}\right), \\
\widetilde{\gamma}_u &:= \frac{1}{\gamma_u} = \frac{\alpha - \mathtt{i}\rho u - \bar{d}_u}{\alpha - \mathtt{i}\rho u + \bar{d}_u}.
\end{aligned}
$$

2.4 The Lewis-Lipton approach

Independently, Lewis in [50] and Lipton (cf. [52] and [51], in the context of FX options) suggest to use a more tractable form of the classical inverse Fourier representation, which corresponds to the re-writing of the payoff of a Call like:

$$
(F_T - K)^+ = F_T - \min(F_T, K)
$$

The abstract pricing formula reads, in a world with no interest rates:

$$
E[(F_T - K)^+] = F_0 - \frac{K}{\pi}\int_0^{+\infty} \frac{\Re\left(\exp((\mathtt{i}u + \frac{1}{2})k)\phi\left(u - \frac{\mathtt{i}}{2}, 0, T\right)\right)}{u^2 + \frac{1}{4}} du
$$

where $k = \log(K)$ and ϕ is the characteristic function of the distribution of the log of the underlying at time T defined in Section 2.3.

One of the main interests of this expression is that there is no singularity in the integrand, unlike the previous approach.

The Lewis-Lipton approach is very often used in practice ([31], [59], [42]). It is also one of the key ingredients in the work of Tehranchi on the asymptotics of implied volatility ([60]).

2.5 Lewis expansion

When the volatility of volatility σ becomes small, the method by integration described in Section 2.1 above looses some accuracy, so Lewis [50] proposed the following approximation. Let $\sigma^2(x)$ denote the implied variance for a log forward moneyness x, we have

$$\sigma^2(x) = I_0 + I_1(x) + I_2(x),$$

where the I_i are given by

$$\begin{aligned}
I_0 &:= \theta + \frac{v_0-\theta}{\kappa T}\left(1-\mathrm{e}^{-\kappa T}\right) \\
I_1(x) &:= \frac{\sigma J_1}{T}\left(\frac{1}{2}+\frac{x}{TI_0}\right) \\
I_2(x) &:= \frac{\sigma^2}{T}\left\{J_3\left(\frac{x^2}{2T^2I_0^2}-\frac{1}{2TI_0}-\frac{1}{8}\right)+J_4\left(\frac{x^2}{T^2I_0^2}+\frac{x}{TI_0}-\frac{4-TI_0}{4TI_0}\right)\right. \\
&\quad\left.+\frac{J_1^2}{2}\left(-\frac{5x^2}{2T^3I_0^3}-\frac{x}{TI_0}+\frac{12+TI_0}{8T^2I_0^2}\right)\right\},
\end{aligned}$$

with

$$\begin{aligned}
J_1 &:= \frac{\rho T}{\kappa}\left(\theta-(v_0-\theta)\,\mathrm{e}^{-\kappa T}\right)+\frac{v_0-2\theta}{\kappa^2}\left(1-\mathrm{e}^{-\kappa T}\right) \\
J_3 &:= \frac{T}{2\kappa^2}\left(\theta-2(v_0-\theta)\,\mathrm{e}^{-\kappa T}\right)+\frac{1-\mathrm{e}^{-\kappa T}}{2\kappa^3}\left(\left(v_0-\frac{\theta}{2}\right)\left(1+\mathrm{e}^{-\kappa T}\right)-2\theta\right) \\
J_4 &:= \frac{\rho^2}{\kappa^3}(v_0-3\theta)+\frac{\rho^2\theta}{\kappa^2}t+\frac{\rho^2}{\kappa^3}(3\theta-v_0)\,\mathrm{e}^{-\kappa t}+\frac{\rho^2 t}{\kappa^2}(2\theta-v_0)\,\mathrm{e}^{-\kappa t}+\frac{\rho^2t^2}{2\kappa}(\theta-v_0)\,\mathrm{e}^{-\kappa t}.
\end{aligned}$$

Remark 6. *Even if σ does not appear anymore in the rescaled formulation of Heston, the limit as $\bar{t}$ tends to zero is not the same as the one in the original Heston when t tends to zero, as the other parameters in the rescaled Heston also depend on σ.*

Small vol of vol asymptotics are also computed in Lipton's book ([51]).

2.6 Simulation of the Heston model

The option pricing formula above it valid for European vanilla options. For exotic options, in particular path-dependent options, one shall need to use either pde methods or Monte Carlo simulations. However, the square-root function in the stochastic differential equation for the variance process in (1) is subtle to handle. As a simple example, let us consider a naive Euler discretisation of the variance process alone between time $t > 0$ and time $t+\Delta t$. We have $v_{t+\Delta t} = (1-\kappa\Delta t)\, v_t + \kappa\theta v_t + \sigma\widetilde{n}\sqrt{v_t\Delta t}$, where $\widetilde{n}$ is a Gaussian random variable with zero mean and unit variance. This clearly implies that given $v_t > 0$, the probability $\mathbb{P}\left(v_{t+\Delta t} < 0\right) = \mathbb{P}\left(\widetilde{n} < -\Big((1-\kappa\Delta t)\, v_t + \kappa\theta\Delta t\Big)/\sqrt{\sigma^2 v_t\Delta t}\right)$ is strictly positive and decreases as the time step Δt tends to zero. When simulating paths of the variance process, even if the Feller condition holds, the process can take negative values, and the square root is then not well-defined anymore. Several "fixes" have been proposed in the literature; they are essentially based on considering, at each time step, some combination of the absolute value (or the positive part) of the variance process. We refer the interested reader to [53] to a precise description of the different Euler schemes that have been developed to solve this issue. In the same spirit, Jaeckel & Kahl [41] have proposed a Milstein scheme to simulate both processes. However their simulation ensures that the variance process remains positive if and only if the condition $4\kappa\theta > \sigma^2$ is satisfied, which is often violated in practice. Unfortunately they do not provide a solution to this, but a fix similar to that for Euler schemes is usually applied in practice.

Even so, according to Broadie & Kaya [12], Euler schemes converge extremely slowly for the simulation of the Heston stochastic volatility model. In [12], they provide an "exact" method for simulating the process, relying on the computation of Bessel functions. Indeed, consider the stock price process following the Heston dynamics (1), then it is straightforward to see that, given $\int_0^t v_s \mathrm{d}s$ and $\int_0^t \sqrt{v_s}\,\mathrm{d}Z_s$, we have

$$\log\left(\frac{F_t}{F_0}\right) \sim \mathcal{N}\left(\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\int_0^t v_s \mathrm{d}s + \frac{\rho}{\sigma}\left(v_t - v_0 - \kappa\theta t\right), \left(1-\rho^2\right)\int_0^t v_s \mathrm{d}s\right),$$

where $\mathcal{N}$ denotes the Gaussian distribution, so that simulating F_t boils down to sampling from the conditional distribution $\left(\int_0^t v_s \mathrm{d}s \mid v_0, v_t\right)$. They do so by numerical inversion of its characteristic function. This scheme has the advantage of being unbiased; however, its numerical complexity makes it difficult to implement efficiently and is henceforth scarcely used in practice. Note that Glasserman & Kim [32] have proposed a way to speed up the computation of the inverse characteristic function using the theory of squared Bessel bridges. Another scheme widely used in practice is the [5] scheme, which carefully handles the conditional distribution of the discretised variance when it becomes negative by choosing a distribution giving weight to the barrier 0 and an additional positive point controlled by the matching of the 1st and 2nd local moments, as usual. More advanced simulation methods have recently been developed in [57] or in [2] and [3], based on cubature methods.

2.7 Calibration

The general approach to the calibration of parametric models, like Heston, is to apply a least-square type procedure either in price or implied volatility. This kind of approach will in general be very sensitive to the choice of the initial point, which will very often in practice drive the selection of the local minima the algorithm will converge to. The various explicit formulae for short or long term asymptotics, conveniently inverted, may come into play to get a pertaining initial point, even if, as we shall see, some care is needed in the *large correlation regime*. It might be useful too to work in rescaled parameters: this will feed the optimisation algorithm with the adequate variable axes.

Such algorithms will typically converge in a few seconds on a standard laptop. An order of magnitude can be gained by using the derivatives of the price, as described for instance in [56].

3 Implied volatility asymptotics

In this section, we present an overview of the many recent results concerning the asymptotic properties of the Heston model, in particular those of the corresponding implied volatility. We first present the results which hold in both correlation regimes, and then those depending on the correlation regime. In the following subsections we shall denote $\sigma_T(x)$ the implied volatility of a European Call option written on the (forward) asset price F_t, with strike $K = F_0\mathrm{e}^x$ maturing at time T. We will also denote $X_t = \log(F_t)$ the log-forward price for convenience. Many results in this section will be based on the knowledge and the properties of the Laplace transform of the model. It hence worth recalling its exact form. For any $t > 0$,we define the moment generating function Λ_t of the Heston model by

$$\Lambda_t(u) := \log \mathbb{E}\left(\mathrm{e}^{u(X_t - x_0)}\right), \quad \text{for all } u \in \mathbb{R} \text{ such that the expectation exists.} \tag{5}$$

3.1 General asymptotic results

3.1.1 Short-maturity behaviour

The short-maturity behaviour of the Heston model has recently attracted quite a few people, with a focus on applications of Varadhan's seminal work [61] on short-time behaviour of diffusion processes. Berestycki et al. [11] showed that, in a fairly general stochastic volatility model, the short-time implied volatility is the viscosity solution to a certain PDE; Durrleman [20] characterised and solved this PDE in the Heston model, and Labordère [36] characterised the small-time behaviour of the implied volatility using the heat kernel expansion on a Riemannian manifold. Finally, in [22], Feng et al. worked out the short-time behaviour of the Heston model in a fast mean-reverting regime using large deviations theory, Alòs & Ewald [4] in a small volatility of volatility regime using Malliavin calculus, and Medvedev & Scaillet obtained the short-time behaviour of general stochastic volatility models by means of asymptotic expansion of the corresponding pricing PDE. We present here the main result of interest in practice for the Heston model, as proved in [23]. Let us define the limiting moment generating function $\underline{\Lambda} : \underline{\mathcal{D}} \to \mathbb{R}$ by $\underline{\Lambda}(u) := \lim_{t\to 0} t\Lambda_t(u/t)$, for all $u \in \underline{\mathcal{D}}$, where $\underline{\mathcal{D}}$ is a closed interval of the real line containing the origin (see [23] for its expression in terms of the Heston parameters). In [23],the authors prove that the function $\underline{\Lambda}$ has the following representation:

$$\underline{\Lambda}(u) = \frac{v_0 u}{\sigma}\left(\sqrt{1-\rho^2}\cot\left(\frac{\rho\sigma u}{2}\sqrt{1-\rho^2}\right) - \rho\right)^{-1}, \qquad \text{for all } u \in \underline{\mathcal{D}}.$$

Let us now define the Fenchel-Legendre transform $\underline{\Lambda}^*$ of $\underline{\Lambda}$ by the variational representation $\underline{\Lambda}^*(x) := \sup\{ux - \underline{\Lambda}(u),\, u \in \underline{\mathcal{D}}\}$, for all $x \in \mathbb{R}$. Then the following theorem holds.

Theorem 7. *Let $F_t \sim \mathcal{H}(F_0, v_0, \kappa, \theta, \sigma, \rho, t)$, then as the maturity t tends to zero,*

$$\lim_{t\to 0}\sigma_t(x) = \begin{cases} \dfrac{|x|}{\sqrt{2\underline{\Lambda}^*(x)}}, & \text{if } x \neq 0 \\ \sqrt{v_0}\left(1 + \dfrac{\rho\sigma x}{4v_0} + \left(1 - \dfrac{5\rho^2}{2}\right)\left(\dfrac{\rho\sigma}{v_0}\right)^2 \dfrac{x^2}{24} + \mathcal{O}\left(x^3\right)\right). & \text{if } x = 0 \end{cases}$$

Remark 8. *The Legendre transform $\underline{\Lambda}^*(x)$ is not available in closed-form here, but can be written as $\underline{\Lambda}^*(x) = xp^*(x) - \underline{\Lambda}(p^*(x))$, where $p^*(x)$ is the unique solution to the equation $\underline{\Lambda}'(p^*(x)) = x$.*

Remark 9. *Note that the ATM result coincides with the formula on page 127 in [50] and section 3.1.2. in [20]. For higher-order terms (in powers of t), see the recent preprint [26]. Adding higher-order terms allows one to calibrate the term structure of the short-maturity implied volatility smile, not the limiting smile only. Forde et al. [26] have proposed a calibration methodology based on these higher-order asymptotics for small maturities in the Heston model, and we refer the interested reader to the corresponding paper for more details on this matter.*

3.2 The *good correlation regime*

We present here asymptotic results in the case $\kappa - \rho\sigma > 0$, i.e. $\alpha > \rho$.

3.2.1 Extreme strikes

In [49], Lee provided the first general result on the asymptotics of the implied volatility, relating the exploding behaviour (to be defined below) of the asset price process to the slope of the smile in the wings. Benaim & Friz ([8] and [9]) sharpened and extended Lee's results by showing under which circumstances the $\limsup$ in Proposition 10 below is a genuine limit. To be able to use this result in practice, one therefore needs the explosion times of the process, which are derived in [6]. In the Heston case, we can provide more precise results, which we state in the following corollary.

Proposition 10. *Fix $t > 0$ and let*

$$\beta_R(t) := \limsup_{x \to +\infty} \frac{\sigma_t^2(x)t}{x}$$

denote the right slope of the total implied variance. Then we have the equality

$$\beta_R(t) = 2 - 4\left(\sqrt{\omega^*(t)\left(\omega^*(t) - 1\right)} - \left(\omega^*(t) - 1\right)\right)$$

where $\omega^*(t) > 1$ is the unique positive solution of $t^*(\omega^*(t)) = \sigma t$ with

$$t^*(\omega) := \frac{4}{\sqrt{-D(\omega)}}\left(\pi \mathbb{1}_{\{a(\omega)<0\}} + \arctan\left(\frac{\sqrt{-D(\omega)}}{a(\omega)}\right)\right)$$

and $a(\omega) := 2(\rho\omega - \alpha)$, $b(\omega) := \omega\left(\omega - 1\right)$, and $D(\omega) := a^2(\omega) - 4b(\omega)$.

The formula given in the above proposition, though not in closed form, is straightforward to solve numerically, as can be seen in Figure 1 below. When t tends to infinity, $\omega^*(t)$ tends to the strictly positive root $\omega_+ > 1$ of $D(\omega)$ and the formula boils down to the following:

Corollary 11. *As t tends to infinity, the following closed-form formula holds,*

$$\beta_R(\infty) = \frac{2}{(1-\rho)}\left(\sqrt{(2\alpha - \rho)^2 + (1 - \rho^2)} - (2\alpha - \rho)\right)$$

Proof. (of Proposition 10 and Corollary 11): both results follow directly from Proposition 3.1 and Corollary 6.2 in [6] as well as [49]. □

The left slope of the asymptotic smile is obtained easily using the symmetry property detailed in Section 1.4. Finally, very recently, Gulisashvili, in [34] and [35] provided sharp asymptotic estimates for the wings of the implied volatility smile under a zero-correlation assumption, thus refining Lee's result. Let us fix a maturity $t > 0$ and consider strikes of the form $F_0 e^x$. Friz et al. [28] have recently improved this wing

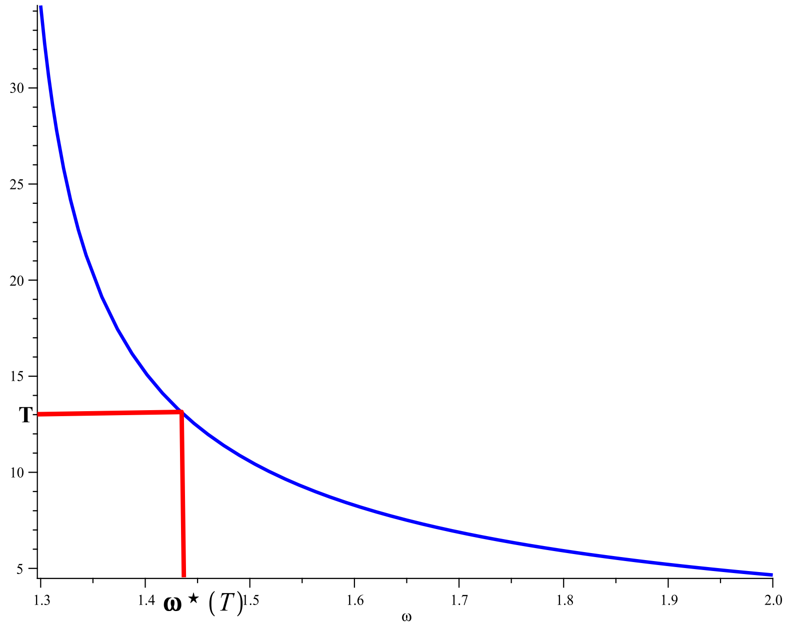

Figure 1: $t^*(\omega)$ for $\alpha = 0.2, \sigma = 0.2, \rho = -0.3$. We illustrate how to numerically determine the optimal ω^* for a given maturity $t = T$.

behaviour in the correlated Heston model. They indeed proved that there exist real constants β_1, β_2 and β_3 such that

$$\sigma_t^2(x)t = \left(\beta_1\sqrt{x} + \beta_2 + \beta_3\frac{\log(x)}{\sqrt{x}} + \mathcal{O}\left(\frac{\phi(x)}{\sqrt{x}}\right)\right)^2, \quad \text{as } x \text{ tends to infinity,} \tag{6}$$

where ϕ is any positive increasing function on $(0, \infty)$ such that $\lim_{x\to\infty} \phi(x) = \infty$. Recall the definition of the moment explosion $t^*(m) := \sup\{t > 0, \mathbb{E}(S_t^m) < \infty\}$. Note that the three constants β_1, β_2 and β_3 are characterised in terms of the critical moment m^* and the critical slope s^* defined by

$$m^* := \sup_{m\geq 1}\left\{\mathbb{E}\left(S_t^m\right) < \infty\right\}, \qquad \text{and} \qquad s^* := -\left.\frac{\partial t^*(m)}{\partial m}\right|_{m=m^*}.$$

3.2.2 Large maturity behaviour

In [50, Chapter 6], Lewis obtained the large-time behaviour of the implied volatility under the Heston model as a second-order polynomial in the forward log-moneyness $X_t = \log(F_t)$. However, his proof exhibits a couple of typos and the resulting formula is partially wrong. In [24], using large deviations theory, Forde & Jacquier rigorously extended (and corrected) Lewis' zeroth order asymptotic result to the case where the strike is of the form $F_0 e^{xT}$ (i.e. is a function of the maturity T). The intuitive reason for this parameterisation is that, as the maturity gets large, the range of possible strikes gets large too. Define the limiting cumulant generating function $\bar{\Lambda} : \overline{\mathcal{D}} \to \mathbb{R}$ by

$$\bar{\Lambda}(u) := \lim_{t\to\infty} t^{-1}\Lambda_t(u) = \psi\sigma \left(\alpha - \rho p - \sqrt{(\alpha-\rho p)^2 - (p^2 - p)}\right),$$

where the moment generating function Λ_t is defined in (5), and where $\overline{\mathcal{D}}$ is a closed interval of the real line containing $[0, 1]$ (see [24] for an explicit representation). We further define its Fenchel-Legendre transform $\bar{\Lambda}^* : \mathbb{R} \to \mathbb{R}$ by $\bar{\Lambda}^*(x) := \sup\left\{ux - \bar{\Lambda}(u),\, u \in \overline{\mathcal{D}}\right\}$. In [24], the authors proved that it has the explicit representation

$$\bar{\Lambda}^*(x)(x) = xp^*(x) - \bar{\Lambda}\left(p^*(x)\right)$$

with

$$p^*(x) := \frac{1-2\alpha\rho}{2(1-\rho^2)} + \frac{(\sigma\psi\rho + x)}{2(1-\rho^2)}\sqrt{\frac{1+4\alpha^2-4\alpha\rho}{x^2+2x\rho\sigma\psi+\sigma^2\psi^2}}.$$

Let us define the function $\sigma_\infty : \mathbb{R} \to \mathbb{R}$ by

$$\sigma_\infty(x) := \sqrt{2}\left(\operatorname{sgn}\left(\frac{\widetilde{\theta}}{2} - x\right)\sqrt{\bar{\Lambda}^*(x) - x} + \operatorname{sgn}\left(x - \frac{\theta}{2}\right)\sqrt{\bar{\Lambda}^*(x)}\right), \tag{7}$$

where we recall that $\widetilde{\theta} := \sigma\psi/\left(\alpha-\rho\right)$.

Theorem 12. *For all real number x the the equality $\lim_{t\to\infty}\sigma_t(x) = \sigma_\infty(x)$ holds.*

Higher-order terms (in t^{-1}) can be obtained to improve the accuracy of the approximation, see [27] for a precise statement and its proof.

Corollary 13. *In the case $x = 0$ in Theorem 12, the large-time at-the-money implied variance is given by*

$$\sigma_\infty^2(0) = \frac{2\psi\sigma}{2\alpha+1}h(\alpha,\rho),$$

where

$$h\left(\alpha,\rho\right) := \frac{8\alpha+4}{1-\rho^2}g(\alpha,\rho), \qquad \text{and} \qquad g\left(\alpha,\rho\right) := \sqrt{\left(\alpha-\frac{\rho}{2}\right)^2 + \frac{1-\rho^2}{4}} - \alpha + \frac{\rho}{2},$$

and where the limit as ρ tends to -1 (or 1) holds since $\lim_{\rho\to-1} h(\alpha,\rho) = 1$.

3.2.3 Long-term ATM slopes and curvature

Using the functions g and h defined in Remark 13 and differentiating the implied variance slopes derived here above, we obtain the long-term ATM implied variance slopes:

$$\lim_{t\to\infty} \partial_x \sigma_t^2(x)t\big|_{x=0} = \frac{2\rho}{2\alpha+1} h(\alpha\rho) \tag{8}$$

Note that these slope results are obtained directly from the large asymptotics of the implied variance, which is not rigorous. Concerning the curvature, the only known result is in [50]. However, it indicates that the curvature is *always* negative, which does not seem reasonable.[a]

3.3 The *large correlation regime*

We derive here some asymptotic results when $\alpha \leq \rho$.

3.3.1 Extreme strikes

We give an equivalent of Proposition 10 holds in the large correlation regime. Recall the definition of the function $\beta_R : [0, \infty) \to \mathbb{R}$ as the right slope of the total implied variance $\sigma_t^2 t$:

$$\beta_R(t) := \limsup_{x\to+\infty} \frac{\sigma_t^2(x)t}{x}.$$

Proposition 14. *For any maturity $t > 0$, the equality*

$$\beta_R(t) = 2 - 4\left(\sqrt{\omega^*(t)\left(\omega^*(t)-1\right)} - \left(\omega^*(t)-1\right)\right)$$

holds, where $\omega^* = \omega^*(t) > 1$ is the unique positive solution of $t^*(\omega^*) = \sigma t$ with

$$t^*(\omega) := \frac{4}{\sqrt{-D(\omega)}} \arctan\left(\frac{\sqrt{-D(\omega)}}{a(\omega)}\right) \mathbb{1}_{\{D(\omega)<0\}} + \frac{2}{\sqrt{D(\omega)}} \log\left(\frac{a(\omega)+\sqrt{D(\omega)}}{a(\omega)-\sqrt{D(\omega)}}\right) \mathbb{1}_{\{D(\omega)\geq 0\}},$$

with

$$a(\omega) := 2\left(\rho\omega - \alpha\right), \quad b(\omega) := \omega\left(\omega - 1\right), \quad D(\omega) := a(\omega)^2 - 4b(\omega).$$

Corollary 15. *The limit of $\beta_R(t)$ as the maturity t tends to infinity is equal to two.*

Proof. As t tends to infinity, the unique root ω^* of $t^*(\omega^*) = t$ tends to 1, so that the corollary follows from Proposition 14 with $\omega^* = 1$. □

[a]This is confirmed by a personal communication with Alan Lewis, whom the authors thank.

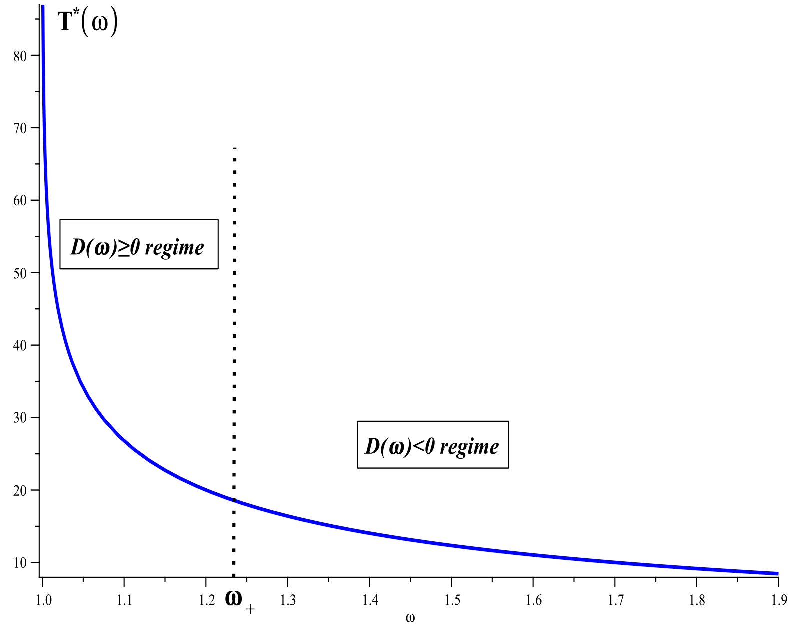

Figure 2: Plot of the function $\omega \mapsto t^*(\omega)$ defined in Proposition 14 for $\alpha = 0.2, \sigma = 0.2, \rho = 0.6$. Note that the function is continuous at the point ω_+ but has two different analytic representations below and above it.

3.3.2 Slippy Slopes

Note that most of the large-time results obtained in Section 3.2.2 rely on the assumption $\kappa - \rho\sigma > 0$ ($\alpha > \rho$). It turns out that, when this condition is not satisfied, we do not know the precise asymptotics. As we mentioned in the introduction, this case is often ruled out in papers, which consider a negative correlation parameter ρ. Although this does make somehow sense on Equity markets, it does not for FX-related derivatives. In [58], the authors prove a sharp upper bound for the slope of the total implied variance (squared volatility multiplied by the maturity). It is then straightforward to see that the known slopes for the implied variance in the Heston model (as presented in the previous sections) violate this upper bound when $\kappa - \rho\sigma \leq 0$.
Recently, [18] and [60] proved some results when this condition fails, the first one studying the moment explosions (not treated in [6] in that case), the second one providing some insights for the long-term implied volatility asymptotics. A somehow similar theorem to Theorem 12 is currently under progress.
The following proposition, together with the subsequent remark, clearly explains in what sense the results derived in the *good correlation regime* do not hold here anymore.

Proposition 16. *(see [58]) Under no-arbitrage conditions, the slope of the total implied variance can not be greater than* 4.

Remark 17. *In (8) it is easy to see that, as ρ tends to one, for all $0 < \alpha < \rho/2$, the quantity $\frac{2\rho}{2\alpha+1}h(\alpha, \rho)$ tends to infinity, which contradicts the above proposition.*

3.4 Application: the SVI connection

We highlight here a concrete example of the usefulness of the aforementioned asymptotic results. In 2004, Jim Gatheral [30] proposed the so-called *Stochastic Volatility Inspired* (SVI) parametric form for the implied (squared) volatility. Let $\sigma^2_{\mathrm{SVI}}(x)$ represent the squared implied volatility evaluated at the log-moneyness $x \in \mathbb{R}$. Gatheral proposed the following parametric form:

$$\sigma^2_{\mathrm{SVI}}(x) = a + b\left(\mathfrak{r}\,(x-m) + \sqrt{(x-m)^2 + \zeta^2}\right), \tag{9}$$

where a and m are real numbers, b and ζ are strictly positive and $\mathfrak{r} \in [-1, 1]$. This parametric form allows for a clear interpretation of each parameter in terms of the shape of the implied volatility smile: a gives the overall level of the variance, m is just a parallel translation parameter along the x axis, ρ determines the degree of symmetry of the graph, ζ determines the smoothness of the at-the-money ($x = 0$) smile and b characterises the angle between small and large x. Even though, as a parametric form, this proposed smile does not guarantee absence of arbitrage, it shows very accurate calibration results to market data.

In Section 3.2.2,we have derived the large-maturity limit σ_∞ (see (7)) for the Heston model defined in (1), for strikes of the form $F_0 \mathrm{e}^{xt}$. Gatheral and Jacquier [31] recently proved the following proposition, already conjectured by Gatheral. Define the two numbers ω_1 and ω_2 as

$$\omega_1 := \frac{4\kappa\theta}{\sigma^2(1-\rho^2)}\left(\sqrt{(2\kappa - \rho\sigma)^2 + \sigma^2(1-\rho^2)} - (2\kappa - \rho\sigma)\right), \quad \text{and} \quad \omega_2 := \frac{\sigma}{\kappa\theta},$$

and consider the following mappings between SVI parameters and Heston parameters:

$$\begin{aligned} a &:= \frac{\omega_1}{2}\left(1-\rho^2\right), \qquad b := \frac{\omega_1\omega_2}{2t}, \qquad \zeta := \frac{\sqrt{1-\rho^2}\,t}{\omega_2}, \\ m &:= -\frac{\rho t}{\omega_2}, \qquad \mathfrak{r} := \rho. \end{aligned} \tag{10}$$

Proposition 18. *With the mapping* (10)*, the equality $\sigma^2_\infty(x) = \sigma^2_{\mathrm{SVI}}(x)$ holds for all $x \in \mathbb{R}$.*

We refer the reader to the corresponding paper [31] for more details about the implications of such a result. This result in particular implies that the SVI parameterisation is free of arbitrage for large enough maturities. One might therefore wonder about fixed maturities. A simple Taylor expansion of the implied volatility as the (log) strike tends to infinity leads to

$$\sigma_{\mathrm{SVI}}(x) = x^{1/2}\sqrt{b(1+\mathfrak{r})} + x^{-1/2}\frac{a - bm(1+\mathfrak{r})}{2\sqrt{b(1+\mathfrak{r})}} + \mathcal{O}\left(x^{-3/2}\right), \quad \text{as } x \text{ tends to infinity.}$$

Therefore the SVI parameterisation (like the Heston model) is linear in the wings for fixed maturities at first order. However, at second order, the Heston model also exhibits a constant term, absent in the SVI formulation, which tends to zero as the maturity tends to infinity (see (6)). For fixed maturities, this implies (i) that this parameterisation is not consistent in the wings with the Heston model and (ii) that some arbitrage can occur.

3.5 Symmetric smiles and the uncorrelated Heston model

In the no-correlation $\rho = 0$ case, the smile is symmetric - the uncorrelated Heston model is invariant by the "Share measure transform". For symmetric models in general, there is an explicit relation between the ATM volatility and the (risk neutral) ATM cumulated probability function (i.e., $P(F_t < F_0)$). In [19], an additional step is performed which gives, for any strike (not only the ATM strike), an explicit formula for the Call/Put price in term of the *Laplace transform* of the cumulated variance $\int_0^T V_u du$. For many stochastic volatility models, this Laplace transform is known explicitely, and semi-closed formulas follow. This encompasses the uncorrelated Heston model. A noteworthy fact is that complex analysis and Fourier transforms are eventually not required in the uncorrelated setting. The article [19] also computes the optimal SVI approximation of the uncorrelated Heston model. This gives very efficient semi-closed approximation formulas for the implied volatility.

4 Beyond the Heston model

4.1 Time-dependent Heston

In order to improve the Heston model—for instance allowing a joint calibration to both equity and volatility derivatives—one could try to make the parameters time-dependent. Let $F_t \sim \mathcal{H}(S_0, v_0, \kappa_t, \theta_t, \sigma_t, \rho_t, t)$. In [13], Buehler showed that in order to be able to fit a variance swap curve consistently, one needs to keep κ_t and the product $\rho_t\sigma_t$ constant. In [25], the authors proposed a method to calibrate a Heston model with time-dependent parameters (all but κ_t) to a given Variance Swap curve. In terms of pricing, Benhamou & al. [10] proposed an analytical approximation for European vanilla options using Malliavin calculus, in a small volatility of volatility regime, which can be seen as an extension of the Lewis asymptotics to the time-dependent case, whereas Elices [21] calibrates a time-dependent Heston model extending the original Fourier-transform approach to the time-dependent case.

However, practitioners are not always satisfied by this way of extending a model to time-dependent parameters and therefore other extensions, keeping a time-homogeneous model, have been proposed. On the technical front, note that if the parameters are time-dependent, one has to watch carefully the branch cuts. The *rotation algorithm* would be very useful for this matter.

4.2 Multifactor Heston models

Another approach in extending the traditional Heston model has been to introduce the so-called double Heston models. There are actually two types of ”Double Heston”, each corresponding to a special approach: Buehler [13] proposed the double Heston as a Heston model where the long-term variance also follows a CIR dynamics. This is consistent with his variance swap curve approach, where each dimension of this ”Multi-Heston” corresponds to a factor of the variance curve. However, there is no closed or semi-closed form formula for the pricing of vanilla options in this framework. The other approach (see [45], [16]) is to introduce several volatility process, independent one from another, each driven by a Brownian motion acting on a separate time scale. This approach is appealing since

- the characteristic function is available in closed-form (hence allowing a semi-closed form formula for vanilla similar to the standard Heston model).
- the induced correlation between the volatility of volatility process and the returns is stochastic, therefore allowing a better fit to the so-called stochastic skew observed on FX markets.
- finally, as it can be represented (in terms of its characteristic function) as the product of two independent Heston, the asymptotic results presented above are easily computable.

Recent papers on this topic include [43] and [7].

4.3 Affine stochastic volatility models with jumps

The method used in [23] and [24] to prove the small and the large-maturity behaviour of the implied volatility in the Heston model is based on the properties of the Laplace transform of the stock price, which is known in closed-form. In [39], the authors extend the large-maturity results to the general class of affine stochastic volatility models with jumps. This class of models include the Heston model, the Heston model with jumps, the Bates model,the Barndorff-Nielsen & Shephard model as well as all exponential

Lévy models. By studying the corresponding Riccati equations and the properties of the moment generating function $\Lambda_t(u) := \log \mathbb{E}\left(e^{uX_t}\right)$, they prove the following theorem under mild conditions generally satisfied by market data. Let $\sigma_t(x)$ denote the implied volatility at maturity t and corresponding to a strike $F_0 e^{xt}$.

Let us define $\Lambda(u) := \lim_{t\to\infty} t^{-1}\Lambda(u)$ and its Fenchel-Legendre transform

$$\Lambda^*(x) := \sup\{ux - \Lambda(x),\, x \in \mathbb{R}\}$$

Let us further define, similarly to (7) the function $\sigma_\infty : \mathbb{R} \to \mathbb{R}_+$ by

$$\sigma_\infty(x) := \sqrt{2}\left(\operatorname{sgn}\left(\Lambda'(1) - x\right)\sqrt{\Lambda^*(x) - x} + \operatorname{sgn}\left(x - \Lambda'(0)\right)\sqrt{\Lambda^*(x)}\right), \quad \text{for all } x \in \mathbb{R},$$

Theorem 19. *The implied volatility σ_t converges pointwise to σ_∞ as t tends to infinity.*

Remark 20. *In [39], the authors prove that the convergence is uniform on any compact subsets of* $\mathbb{R} \setminus \{\Lambda'(0), \Lambda'(1)\}$.

Conclusion

Although known and used for quite some time both by practitioners and academics, a rather large number of papers recently exhibited some lesser known (or unknown so far) features of the Heston model, therefore greatly improving our understanding, and allowing for robust and more efficient pricing and calibration algorithms, even if some theoretical issues still remain, notably in the *large correlation* regime $\kappa - \rho\sigma \leq 0$.

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