

Robust calibration and arbitrage-free interpolation of SSVI slices

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We describe a robust calibration algorithm of a set of SSVI slices (i.e. a set of 3 SSVI parameters θ, ρ, φ attached to each option maturity available on the market), which grants that these slices are free of Butterfly and Calendar-Spread arbitrage. Given such a set of consistent SSVI parameters, we show that the most natural interpolation/extrapolation *of the parameters* provides a full continuous volatility surface free of arbitrage. The numerical implementation is straightforward, robust and quick, yielding an effective, parsimonious solution to the smile problem, which has the potential to become a benchmark one.

We thank Antoine Jacquier and Stefano De Marco for useful discussions and remarks. All remaining errors are ours.

1. Introduction: the xSSVI family

Gatheral initiated in 2004 the *Stochastic Volatility Inspired* parametric model for the volatility smile, a 5-parameter formula for the total variance at a given maturity. Gatheral's inspiration was of geometric nature, in relation to Roger Lee's moment formula and also to his experience of smiles produced by stochastic volatility models like Heston. SVI fits remarkably well in practice, and it is even difficult to find circumstances in which SVI fails (Fabien Le Floch provides such an example on his blog [4]). Despite its simplicity, the calibration of SVI is not straightforward, and Zeliade has a whitepaper (cf [1]) with a re-parameterization trick which robustifies a lot the process (more detailed calculations are available in Stefano De Marco's PhD Thesis, [3]).

SVI has 2 main missing features: it does not model the whole volatility surface, and there are no known conditions on SVI parameters which grant absence of arbitrage (even tractable sufficient conditions).

Then comes SSVI: many teams worked on producing an SVI-like model for the whole volatility surface in years around 2010, and the only successful one was the Jim Gatheral and Antoine Jacquier pair, who designed the *Surface SVI* model which has the 2 features SVI missed (cf [5]). SSVI is (this may seem natural) parameterized by the ATM (Forward) total variance curve θ_t , so it will automatically fit perfectly the ATMF point, a *constant* correlation parameter ρ (which should play the role of the leverage parameter), and a curvature curve φ :

$$w(k, \theta_t) = \frac{\theta_t}{2} \left(1 + \rho\varphi(\theta_t)k + \sqrt{(\varphi(\theta_t)k + \rho)^2 + (1 - \rho^2)} \right) \quad (1)$$

Each smile has so only 3 parameters, and explicit and tractable *sufficient* conditions have been obtained by Gatheral and Jacquier to preclude Butterfly arbitrage:

$$\theta_t \varphi(\theta_t) \leq \frac{4}{1 + |\rho|} \quad (2)$$

$$\theta_t \varphi(\theta_t)^2 \leq \frac{4}{1 + |\rho|} \quad (3)$$

Moreover, necessary and sufficient no Calendar Spread conditions are provided. SSVI works reasonably well in practice, and its calibration is easier than SVI. Yet the fact to keep the correlation constant across maturities depreciates the fit quality. Sebas Hendriks (student of Kees Osterlee at the University of Delft, during his master internship at Zeliade) and Claude Martini tackled this issue (cf [2]): they managed to obtain

simple necessary and sufficient conditions for the consistency of 2 SSVI *slices* attached to different maturities; such conditions are not available for SVI smiles. By a SSVI slice we mean the SSVI parameterization at a given maturity, with its own correlation parameter, possibly depending on the maturity. Conditions for the absence of Calendar Spread arbitrage in continuous time follow, and the corresponding *extended SSVI* (eSSVI) model extends SSVI with a maturity-dependent $\rho(\theta_t)$. An explicit representation formula for the correlation is also obtained, which allows to produce easily concrete low-dimensional parametrization for the correlation curve. Some power law type examples are provided.

The effective calibration of eSSVI has not been investigated yet, this is the purpose of this note. We proceed in 2 steps, each of them having some interest on its own:

1. We calibrate SSVI (equivalently, eSSVI) slices to the available maturities on the market in a way which grants the absence of Butterfly and of Calendar Spread arbitrage, making use of a very robust calibration algorithm, which *does not use any blackbox optimizer* beyond a one-dimensional Brent algorithm.
2. We show that the most naive interpolation/extrapolation scheme of the *slice parameters* is arbitrage free. This is a remarkable property of eSSVI.

We obtain therefore a continuous time arbitrage-free eSSVI model calibrated to the market. We discuss in the conclusion the virtues of this scheme, that we consider as the quickest and cheapest way (so far) to solve (not perfectly though, but with a sufficient accuracy in many situations but the most demanding ones) the smile problem.

2. Anchored eSSVI slices with no Butterfly arbitrage

The key ingredient in our algorithm is a re-parametrization of a SSVI slice, which constrains the slice to go through the data point (k^*, θ^*) closest to the ATM (Forward), where k denote log-forward moneyness and θ the total implied variance. Whence the word *anchored* in the section title. This re-parameterization assumes that the data in this range are very reliable, which is certainly true for not too-long term options on indexes at least.

So θ will be expressed in terms of the parameters ρ, φ and this new data-driven (k^*, θ^*) pair. At first order this amounts simply to $\theta = \theta^* - \rho\theta\varphi k^*$. We also substitute a new parameter ψ to the product $\theta\varphi$, so that eventually our anchored smiles (anchored to (k^*, θ^*)) are parameterized by the pair (ρ, ψ) .

This anchor trick can be seen as a refinement of Gatheral and Jacquier initial idea to read the ATM Forward volatility on the market (and so, take it as a parameter): it avoids a pre processing step of the market data which computes θ by interpolation from the available bracketing strikes, which brings some noise, or the handling of θ as an additional parameter to calibrate, which adds a dimension.

Note that we could try to anchor to more than one point, yet this is likely to put too many constraints on the parameters, especially for large maturities.

2.1 No Butterfly arbitrage

Translating the short term no butterfly constraint reads:

$$\psi \leq 2\sqrt{\frac{\theta}{1+|\rho|}}$$

or yet $\psi^2 \leq \frac{4}{1+|\rho|}(\theta^* - \rho\psi k^*)$ which is equivalent to the explicit bound

$$\psi \leq \psi_+(\rho, k^*, \theta^*)$$

where $\psi_+(\rho, k^*, \theta^*) = \frac{-2\rho k^*}{(1+|\rho|)} + \sqrt{\frac{4\rho^2(k^*)^2}{(1+|\rho|)^2} + \frac{4\theta^*}{(1+|\rho|)}}$.

In other words, *all the no-butterfly arbitrage (in the sense that they satisfy the Gatheral Jacquier bounds) eSSVI slices anchored at (k^*, θ^*)* are parameterized by the SSVI formula, where θ is replaced by its expression in terms of (k^*, θ^*) , $\rho \in]-1, 1[$ and $0 < \psi < \min(\psi_+(\rho, k^*, \theta^*), \frac{4}{1+|\rho|})$.

Also θ should be non negative, so that the constraint $\psi < \frac{\theta^*}{\rho k^*}$ should be enforced when active.

Note in passing that if k^* is assumed to be small, the following expression for the SSVI total variance can be used:

$$\frac{1}{2}(\theta^* + \rho\psi(k - k^*) + \sqrt{(\psi(k - k^*) + \rho\theta^*)^2 + (1 - \rho^2)(\theta^*)^2})$$

3. Granting no Calendar-Spread arbitrage across slices

Thanks to the result in Hendriks-Martini [2] we have necessary and sufficient conditions for this. Let $(\theta_i, \rho_i, \varphi_i)_{1 \leq i \leq N}$ a set of (e)SSVI slice parameters corresponding to increasing time to maturities $0 < T_1 < \dots < T_N$ with $N > 1$.

Then θ_i and ψ_i should be non-decreasing, and the condition:

$$\left| \frac{\rho_{i+1}\psi_{i+1} - \rho_i\psi_i}{\psi_{i+1} - \psi_i} \right| \leq 1$$

should hold.

4. Going forward calibration

Let us re-formulate those conditions in the setting where we calibrate the slices *going forward*: we start by calibrating $(\theta_1, \rho_1, \varphi_1)$, so catering only for the absence of Butterfly arbitrage for this initial slice.

The slices are then built in the following way, where we denote by $\underline{\theta}, \underline{\psi}, \underline{\rho}\underline{\psi}$ the corresponding quantities associated to the *previous slice*, and by (θ, ρ, ψ) the SSVI parameters for the current slice.

The absence of Calendar Spread between the two slices is granted by the conditions $\theta > \underline{\theta}$, $\psi > \underline{\psi}$ and the last condition that reads $-(\psi - \underline{\psi}) \leq \rho\psi - \underline{\rho}\underline{\psi} \leq (\psi - \underline{\psi})$, which amounts to $\psi \geq \psi_-(\rho)$ where

$$\psi_-(\rho) := \max\left(\frac{\psi - \underline{\rho}\underline{\psi}}{1 - \rho}, \frac{\psi + \underline{\rho}\underline{\psi}}{1 + \rho}\right)$$

So we get again bound type conditions on ψ given ρ and the previous slice parameters. Only the first condition $\theta > \underline{\theta}$ is to be investigated: by substituting $\theta = \theta^* - \rho\theta\varphi k^*$, it also amounts to a bound type constraint $\psi > \hat{\psi}$ or $\psi < \hat{\psi}$ depending on the sign of ρk^* , where

$$\hat{\psi} := \frac{\theta^* - \underline{\theta}}{\rho k^*}$$

The particular case $\rho = 0$ yields the constraint $\theta^* > \underline{\theta}$, which is directly checked on the market data at the current slice.

5. Implementation

5.1 Algorithm

Putting all the constraints together, for a given ρ , we get a set of two-sided bound type constraints for ψ (which is positive) possibly empty, which grants simultaneously no Butterfly arbitrage and no Calendar Spread arbitrage with the previous slice. Given any fit objective function (a good choice is the L^1 norm of the price differences between the eSSVI price and the market price), we face for each slice a 2 dimensional function in ψ, ρ . A very effective way to solve the minimization problem is to proceed as follows:

1. Sample ρ in the interval $] -1, 1[$.
2. For each sampled ρ use a Brent algorithm to find the point ψ , satisfying the constraints, at which the minimum of the objective function is obtained.
3. Pick up the minimum over all the ρ .
4. Repeat the procedure on a smaller interval centered on the optimal ρ found before.

This is very naive, yet very robust, quick and effective. Moreover the minimization can be split ρ -wise on different cores.

The global algorithm consists in calibrating the first slice, and then the subsequent slices with the constraints produced by the calibrated parameters attached to the previous slice.

5.2 Comments

5.2.1 Choice of the initial slice

It is natural to start from the short-term slice. Very often there is a lot of curvature at short maturities, and the close-to-ATM option price is roughly proportional to the ATM volatility, so that there will be enough meaningful data to calibrate the initial pair ρ, ψ . There might be issues though for very short maturity where the market will convey only information on θ (or in this case, equivalently, θ^*) and not on ρ and φ . Starting from the long term end is more daring, since data is in general less reliable, and there might be much less curvature due to the fact that implied volatility smiles flatten for large maturities. Depending on the underlying (in terms of liquidity) and the dataset (in terms of available time-to-maturities), different strategies may be considered, including intermediate ones where the initial slice is a mid term one and the algorithm goes in both directions. In this case the algorithm should be tweaked for the going-backward part, with computing the upper constraints instead of the lower ones.

5.2.2 Data consistency

It may also happen that the feasible dataset for a given slice is empty (though this never happened in our tests), due either to a too extreme calibrated smile at a previous maturity or dubious data at the current one. In particular if the (k^*, θ^*) data point is not reliable, this algorithm should not be run.

5.2.3 Robustness

This is the most appealing feature of this algorithm: besides the number of ρ sampling points, there is no starting point nor numerical parameter to set and tweak, the algorithm is extremely robust and in this respect can be put safely in production. 20 points is enough according to us to grant a calibration within the bid ask in general.

5.2.4 Stability of the calibrated parameters

At each slice, the free parameters correspond to the ATM Forward slope and curvature. The fact that only 2 degrees of freedom remain after the anchoring trick brings more stability than the traditional 3-dimensional criterion. Lastly, the no Calendar Spread constraints ensure a built-in consistency across slices which also brings a lot of stability to the calibrated values.

5.2.5 Speed

Without any parallelization, a Python implementation for 12 maturities and an average of 98 options per maturity (the number of options per maturity varies between 68 and 184) takes 1.2 seconds on a *Intel E5-2673 v3* processor.

On a more recent processor, as the *Intel Xeon E7-8890 v3*, by parallelizing the computation of the function ρ -wise the execution time should be cut down to 0.1 second or less. A C# implementation could also reduce the computation time by a factor 5 (quite a conservative estimate) to a final execution time of 0.01 seconds or less.

6. Results

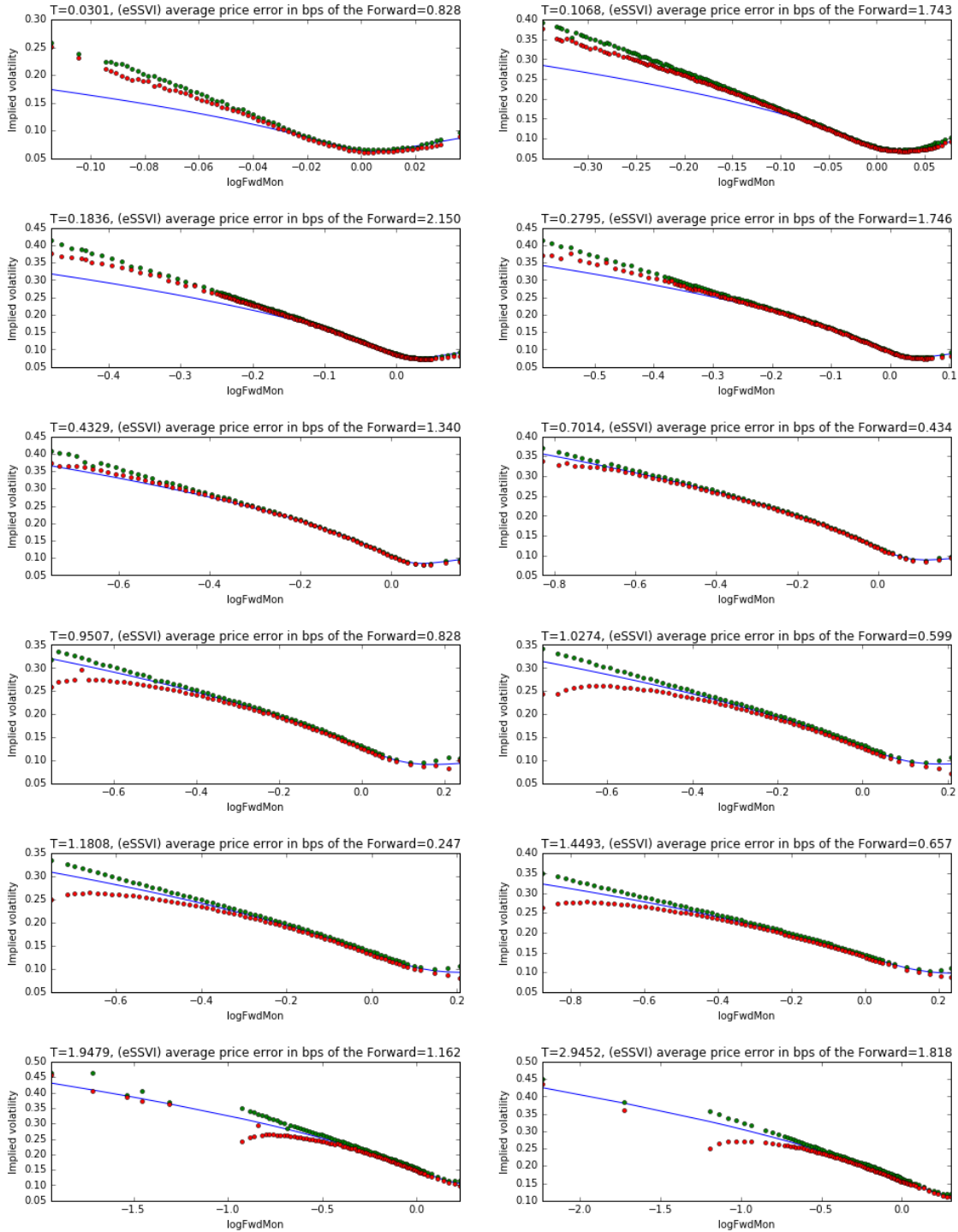
We did numerous tests on several non public data sets by our clients, on Equity Index and Equity Stock options. The algorithm performed systematically very well, with a typical average option price error below 4 bips of the underlying value. We display here some results obtained on end-of-day SPX option quotes (data acquired from the CBOE, <https://datashop.cboe.com/option-quotes>) on January 8th, 2018.

6.1 Data processing

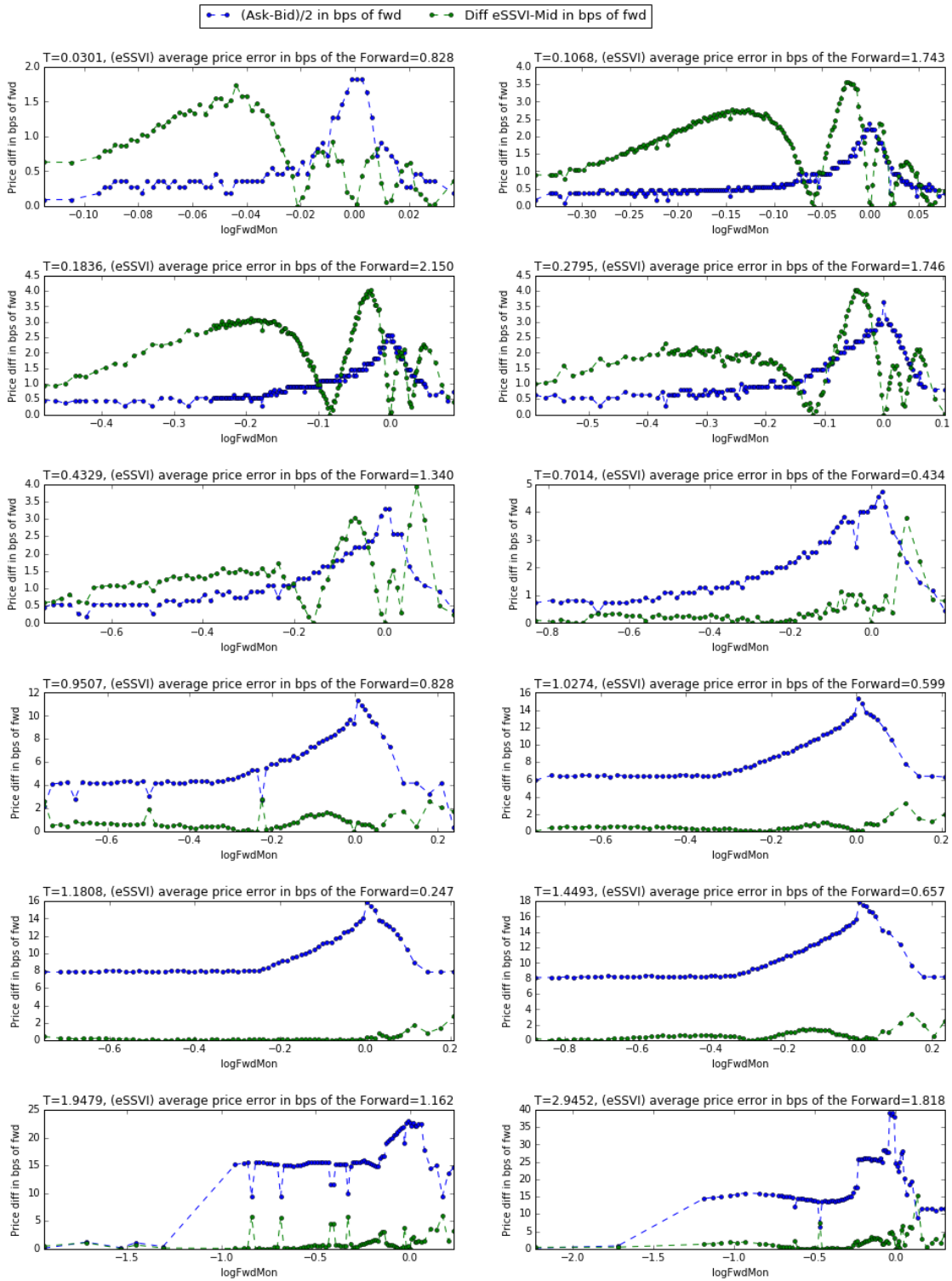
We infer the Forward and Discount Factor at each available maturity by robust linear regression leveraging the Put-Call-Parity for mid prices. Then we select OTM options and filter out prices which are below 2 ticks (the tick being 0.05 for SPX options). Implied volatility is computed using Jaeckel *rational* algorithm.

Options Implied Vol Fit (SPX)

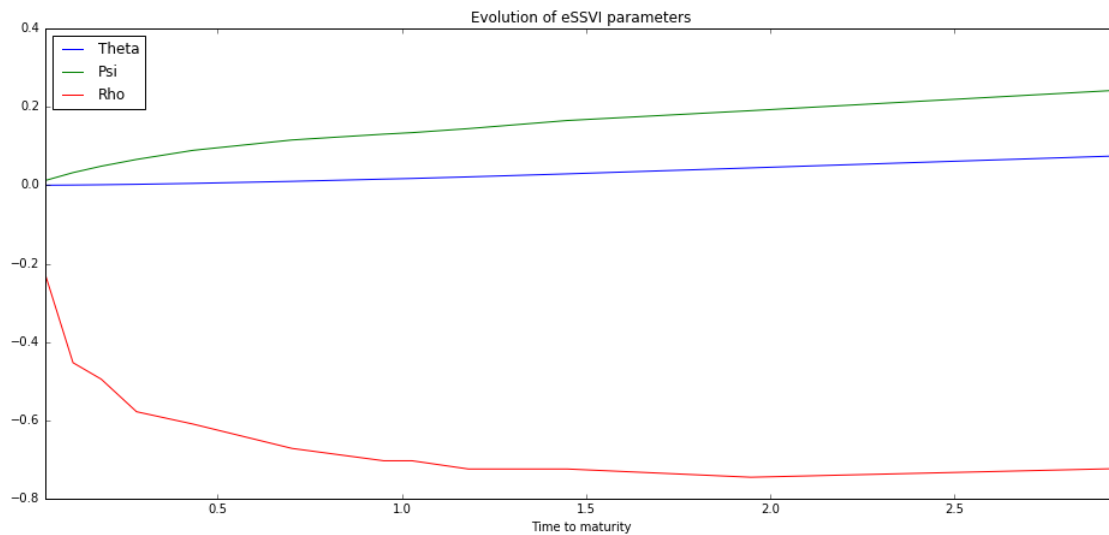
— volModel ● volAsk ● volBid



Market-Model difference in basi points of the Forward (SPX)



	Theta	Psi	Rho	ATM vol (pct)	Phi
Time to maturity					
0.030137	0.0001	0.012	-0.224	6.4	96.33
0.106849	0.0006	0.032	-0.453	7.8	50.22
0.183562	0.0014	0.049	-0.495	8.6	35.82
0.279452	0.0025	0.066	-0.578	9.4	26.66
0.432877	0.0049	0.089	-0.610	10.6	18.38
0.701370	0.0100	0.116	-0.672	12.0	11.55
0.950685	0.0158	0.131	-0.704	12.9	8.28
1.027397	0.0174	0.134	-0.704	13.0	7.73
1.180822	0.0215	0.145	-0.725	13.5	6.75
1.449315	0.0292	0.165	-0.725	14.2	5.68
1.947945	0.0444	0.191	-0.746	15.1	4.29
2.945205	0.0750	0.243	-0.724	16.0	3.24



6.2 Comments

The implied vol fits are well within the bid-ask, except for the 3/4 shorter maturities for the left-end wing. Those visual vol discrepancies translate though in very small errors for the price, as confirmed by the price error plots, below 4 bips of the Forward (i.e., $4 \times 10^{-4} \times$ the Forward value). One should consider also the traded volumes, which are almost zero in this range of strike for those maturities.

Otherwise the fit in price is excellent across all the maturities. The shape of the calibrated correlation is typical, and shows the benefit of eSSVI versus the classical SSVI for calibrating simultaneously the medium/long entries and the shorter ones. Lastly, the calibrated parameters evolve smoothly with the time-to-maturity.

7. Arbitrage-free interpolation

Our starting point in this section is a set of SSVI slice parameters $(\theta_i, \psi_i, \rho_i)_{1 \leq i \leq n}$, attached to maturities $0 < T_1 < \dots < T_n$ and such that:

- Each slice is free of Butterfly arbitrage;
- There is no Calendar Spread arbitrage between any 2 consecutive slices.

Note that the last property amounts to the fact that the total variance smile attached to the longer maturity lies strictly above the smile attached to the lower one. It follows from this geometrical point of view that this property is transitive, so that there is no Calendar Spread for all the slices globally.

It is required in practice to get a continuous arbitrage free volatility surface from these slices. We show below that the natural interpolation and extrapolation of the eSSVI parameters provides a *continuous eSSVI surface* which is indeed arbitrage free. This is a very nice property of the eSSVI parameterization, and note that it is by no way built-in or automatic.

7.1 Interpolation scheme

We describe the interpolation scheme between 2 consecutive slices, which we denote by $(\theta_i, \psi_i, \rho_i)$ and $(\theta_{i+1}, \psi_{i+1}, \rho_{i+1})$.

The no arbitrage conditions read:

1. $\theta_{i+1} > \theta_i$
2. $\psi_{i+1} \geq \psi_i$
3. $\psi_j \leq \min\left(\frac{4}{1+|\rho_j|}, 2\sqrt{\frac{\theta_j}{1+|\rho_j|}}\right)$ for $j = i, i + 1$
4. $\left| \frac{\rho_{i+1}\psi_{i+1} - \rho_i\psi_i}{\psi_{i+1} - \psi_i} \right| \leq 1$

For $\lambda \in [0, 1]$ we define the following interpolation scheme:

- $\theta_\lambda = (1 - \lambda)\theta_i + \lambda\theta_{i+1}$;
- $\psi_\lambda = (1 - \lambda)\psi_i + \lambda\psi_{i+1}$;
- $\rho_\lambda\psi_\lambda = (1 - \lambda)\rho_i\psi_i + \lambda\rho_{i+1}\psi_{i+1}$.

Each such slice will be attached to a maturity t such that $\lambda = \frac{t - T_i}{T_{i+1} - T_i}$.

7.1.1 Calendar Spread arbitrage

Since θ_λ and ψ_λ interpolate linearly between ordered quantities, we will have $\theta_\lambda < \theta_\mu$ and $\psi_\lambda < \psi_\mu$ for $0 \leq \lambda < \mu \leq 1$. In the same way since $\rho_\lambda\psi_\lambda - \rho_\mu\psi_\mu = (\lambda - \mu)(\rho_{i+1}\psi_{i+1} - \rho_i\psi_i)$ and $\psi_\lambda - \psi_\mu = (\lambda - \mu)(\psi_{i+1} - \psi_i)$ condition 4 is satisfied also. So there is no calendar spread arbitrage in between 2 interpolated slices within the same bucket ($i; i + 1$).

By the transitivity property above we deduce that there is no arbitrage between 2 interpolated slices in different buckets.

7.1.2 Butterfly arbitrage

To alleviate notations we will write the proof for the 1st maturity bucket.

We start by checking that $\psi_\lambda < \frac{4}{1+|\rho_\lambda|}$ holds. Since the condition is verified for $\lambda = 0, 1$ it suffices to show that the derivative of $f(\lambda) = \psi_\lambda(1 + |\rho_\lambda|) = \psi_\lambda + |\rho_\lambda\psi_\lambda|$ has constant sign. Since $f'(\lambda) = \psi_2 - \psi_1 + (\rho_2\psi_2 - \rho_1\psi_1)\text{sign}(\rho_\lambda)$ and $\psi_\lambda, \rho_\lambda$ satisfy condition 2. and 4., we conclude that $f' > 0$.

Moreover, whenever ρ_1 and ρ_2 have the same sign, the function f is linear, while if the sign of ρ changes, the function f is piecewise linear.

We now check that the condition $\psi_\lambda < 2\sqrt{\frac{\theta_\lambda}{1+|\rho_\lambda|}}$ holds. This is equivalent to requiring that

$$\psi_\lambda(\psi_\lambda + |\rho_\lambda\psi_\lambda|) < 4\theta_\lambda$$

We will start by considering the case where ρ_1 and ρ_2 have the same sign:

In this case we can rewrite the previous equation as

$$(\psi_1 + \lambda(\psi_2 - \psi_1))(a + b\lambda) < 4(\theta_1 + \lambda(\theta_2 - \theta_1)),$$

where

$$a = \psi_1(1 + |\rho_1|) \quad \text{and} \quad b = \psi_2 - \psi_1 + |\rho_2|\psi_2 - |\rho_1|\psi_1 > 0.$$

Observe that the LHS is a convex function since $b > 0$, so it lies below its chord on $[0, 1]$, which in turn lies below the RHS since the requirement is fulfilled for $\lambda = 0, 1$.

We are now left with the case where ρ_0 and ρ_1 have different signs:

In this case there is a unique λ^* such that $\rho_{\lambda^*} = 0$. Condition 3. reduces, for λ^* , to

$$\psi_{\lambda^*} < \min(4, 2\sqrt{\theta_{\lambda^*}}).$$

Since $\psi_i < 4$, $\psi_i < 2\sqrt{\theta_i}$ $i = 0, 1$, and since the square root is a concave function, the condition above is satisfied at λ^* .

Since f is linear on each interval $[0, \lambda^*]$ and $[\lambda^*, 1]$ we can apply the reasoning for ρ with constant sign on the two intervals $[0, \lambda^*]$ and $[\lambda^*, 1]$ to conclude that the non-arbitrage conditions are verified also in the case in which ρ_1 and ρ_2 have different signs.

7.2 Short term extrapolation

How to extrapolate to the time bucket $]0, T_1[$?

Since the total variance should goes to zero in any reasonable model as the time to maturity goes to zero, it is necessary in a continuous time eSSVI from the no arbitrage condition 3. that ψ goes to zero as well. Therefore the simplest short term extrapolation scheme is as follows:

- $\theta_t = \lambda\theta_1$;
- $\psi_t = \lambda\psi_1$;

- $\rho_t = \rho_1$.

Here $\lambda = \frac{t}{T_1}$.

Then conditions 1., 2. and 4. are readily checked. Conditions 3. reads in turn:

$$\lambda\psi_1 < \min\left(\frac{4}{1+|\rho_1|}, 2\sqrt{\frac{\lambda\theta_1}{1+|\rho_1|}}\right)$$

Now $\lambda\psi_1 < \psi_1 < \frac{4}{1+|\rho_1|}$ and $\lambda\psi_1 < 2\sqrt{\frac{\lambda\theta_1}{1+|\rho_1|}}$ follows from the fact that $\sqrt{\lambda}\psi_1 < \psi_1 < 2\sqrt{\frac{\theta_1}{1+|\rho_1|}}$ for $\lambda < 1$.

Lastly the absence of Calendar Spread arbitrage between 2 slices within the first maturity bucket is shown exactly as above, and between a slice in the first maturity bucket and another one after T_1 by transitivity.

7.3 Long term extrapolation

To extrapolate beyond T_N , pick up any continuous increasing function $u(t)$ on $[T_N, \infty[$ such that $u(T_N) = 0$, and set:

- $\theta_t = \theta_N + u(t)$;
- $\psi_t = \psi_N$;
- $\rho_t = \rho_N$.

Then conditions 1. to 4. are readily checked. In the same way there is no calendar spread arbitrage between 2 slices living beyond T_N . With the same transitivity argument as before, there is no calendar spread arbitrage between one such slice and a slice living below T_N .

8. Conclusion

We have designed a novel calibration algorithm of the eSSVI model, which relies on the forward slice-by-slice calibration of SSVI slices constrained to go exactly through the data point closest to the Forward, computing explicitly the no Butterfly and no Calendar Spread constraints. The naive piecewise interpolation/extrapolation of the slice parameters is shown to be also free of arbitrage. All in all we have a simple, quick and robust calibration algorithm of the volatility surface, which fits very well except maybe in the more demanding (tight market-making) situations. Moreover it is straightforward to *store* and *re-use* the calibrated parameters $(\theta_i, \rho_i, \psi_i)_{1 < i < N}$ alongside the market parameters $(T_i, F_i, DF_i)_{1 < i < N}$ (where F denoted the Forward and DF the Discount Factor) to parsimoniously serialize the whole volatility surface, which is very useful for constituting *histories* of volatility surfaces, e.g. for risk purposes.

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